Arbitrary permutations without repetitions among N digits in *base* r are congruent modulo $(r-1)$

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Proof:

Let be $\mathbf{P}_0(r) = \sum_{\psi=1}^{N} C_\psi r^{(N-\psi)}$ and $\mathbf{P}_1(r) = \sum_{\psi=1}^{N} \ddot{C}_\psi r^{(N-\psi)}$ the polynomial representations for two permutations without repetitions, built both from a subset of N digits[1] in *base* r (by the context it is provided that $N \leq r$). One of them P_0 is arbitrary, and the another P_1 is obtained by transposition of two digits in P_0; Let be $i \neq j$ the indices for the pair (C_i, C_j) of digits in P_0 to be transposed. The pair transposition operation might be described as an assignation rule among the digits in P_0 and P_1 interpreted as the C and $\ddot{C}$ coefficients:

$$\ddot{C}_\psi = \begin{cases} C_\psi & \text{if and only if } i \neq \psi \neq j \\ C_j & \text{if and only if } \psi = i \\ C_i & \text{if and only if } \psi = j \end{cases}$$

Since it exist some h such that $1 \leq h = |i-j| \leq (N-1)$, let us call k to the smallest index and the other will be $k+h$. Then, the difference $|\Delta\mathbf{P}_{10}(r)| = |\mathbf{P}_1(r) - \mathbf{P}_0(r)|$ is:

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[1]Not necessarily consecutive among them.

$$
\begin{aligned}
\left|\Delta\mathbf{P}_{10}(r)\right| &= \left|\mathbf{P}_1(r)-\mathbf{P}_0(r)\right| \\
\left|\mathbf{P}_1(r)-\mathbf{P}_0(r)\right| &= \left|\sum_{\psi=1}^{N}\ddot{C}_{\psi} r^{(N-\psi)} - \sum_{\psi=1}^{N} C_{\psi} r^{(N-\psi)}\right| \\
&= \left|\sum_{\psi=1}^{N}\left[\ddot{C}_{\psi}-C_{\psi}\right]\left[r^{(N-\psi)}\right]\right| \\
&= \left|\sum_{i\neq\psi\neq j}\left[\ddot{C}_{\psi}-C_{\psi}\right]\left[r^{(N-\psi)}\right]+\cdots\right| \\
&= \left|\sum_{i\neq\psi\neq j}\cancel{\left[C_{\psi}-C_{\psi}\right]}\left[r^{(N-\psi)}\right]+\cdots\right| \\
&= \left|0+\cdots\right| \\
&= \left|\sum_{\psi=i,\psi=j}\left[\ddot{C}_{\psi}-C_{\psi}\right]\left[r^{(N-\psi)}\right]\right| \\
&= \left|\left[\ddot{C}_{i}-C_{i}\right]\left[r^{(N-i)}\right]+\left[\ddot{C}_{j}-C_{j}\right]\left[r^{(N-j)}\right]\right| \\
&= \left|\left[C_{j}-C_{i}\right]\left[r^{(N-i)}\right]+\left[C_{i}-C_{j}\right]\left[r^{(N-j)}\right]\right| \\
&= \left|\left[C_{j}-C_{i}\right]\left[r^{(N-i)}\right]-\left[C_{j}-C_{i}\right]\left[r^{(N-j)}\right]\right| \\
&= \left|\left[C_{j}-C_{i}\right]\left[r^{(N-i)}-r^{(N-j)}\right]\right| \\
&= \left|C_{j}-C_{i}\right|\left|r^{(N-i)}-r^{(N-j)}\right| \\
&= \left(C_{k+h}-C_k\right)\left|\left(r^{N-(k+h)}-r^{N-k}\right)\right| \\
&= \left(C_{k+h}-C_k\right)\left|\left(r^{(N-k)}r^{-h}-r^{(N-k)}\right)\right| \\
&= \left(C_{k+h}-C_k\right)\left|r^{(N-k)}\left(r^{-h}-1\right)\right| \\
&= \left(C_{k+h}-C_k\right)\left|r^{[N-(k+h)]}\left[-\sum_{\omega=0}^{(h-1)} r^{\omega}\right](r-1)\right| \\
&= \left(C_{k+h}-C_k\right) r^{[N-(k+h)]}\left(\sum_{\omega=0}^{(h-1)} r^{\omega}\right)(r-1)
\end{aligned}
$$

Clearly "*quo erat demonstrandum*". [2]Anyway:

$$
\left|\Delta\mathbf{P}_{10}(r)\right| = \left(C_{k+h}-C_k\right) r^{[N-(k+h)]}\left(r^{h}-1\right)
$$

[2]Additionally: More complex patterns of permutations might be interpreted as if they were composed by finite successions of pair transpositions.

APPENDIX

Theorem:

The concatenation of digits in ascending order from left to right for a given base, is the smallest possible permutation without repetitions that can be built from those digits.

Proof:

From the main proof let us take the expression for the difference between $\mathbf{P}_0(r)$ and $\mathbf{P}_1(r)$, but now instead of choosing $\mathbf{P}_0$ arbitrarily, *let it be* the concatenation in ascending order from left to right. While the digits (the coefficients C) are placed in ascending order from left to right, the associated powers of r in $\mathbf{P}_0(r)$ are placed in descending order, then any transposition will leave placed at $\mathbf{P}_1(r)$ a greater coefficient for the greatest power of r involved by the transposition. The described condition is enough in order to ensure for every $\mathbf{P}_1(r)$ obtained[3] from $\mathbf{P}_0(r)$ that $\mathbf{P}_0(r) \leq \mathbf{P}_1(r)$; *Q.E.D.*

Theorem:

The concatenation of digits in descending order from left to right for a given base, is the greatest possible permutation without repetitions that can be built from those digits.

Proof:

From the main proof let us take the expression for the difference between $\mathbf{P}_0(r)$ and $\mathbf{P}_1(r)$, but now instead of choosing $\mathbf{P}_0$ arbitrarily, *let it be* the concatenation in descending order from left to right. Both sets, the digits (the coefficients C) and the associated powers of r in $\mathbf{P}_0(r)$ are placed in descending order, then any transposition will leave placed at $\mathbf{P}_1(r)$ an smaller coefficient for the greatest power of r involved by the transposition. The described condition is enough in order to ensure for every $\mathbf{P}_1(r)$ obtained from $\mathbf{P}_0(r)$ that $\mathbf{P}_0(r) \geq \mathbf{P}_1(r)$; *Q.E.D.*

[3]Notice that by successive transpositions you might obtain again $\mathbf{P}_0(r)$.

$\mathbb{A}\ \mathbb{P}\ \mathbb{P}\ \mathbb{E}\ \mathbb{N}\ \mathbb{D}\ \mathbb{I}\ \mathbb{X}$ ([4])

The initial theorem now justified using tensors. Given N a natural number: $\left(x^N - 1\right)$ is divisible by $(x-1)$

There exist a triangular matrix usually called "nonstrict upper unitriangular" whose elements are all either zero or one, being zero those elements below the diagonal. When it operates on the column vector representation for the polynomial $\left(x^N - 1\right)$ built by setting to zero all the components excepting two: the first and the last (where it is assumed that such vector reads from up to down the monomials in descending order by degree), the column vector resulting from the described transformation is compatible with the interpretation of the polynomial $\sum_{k=0}^{(N-1)} x^k$ or represents it under the convention previously described, as shown below:

$$(-1)\begin{pmatrix} 1 & 1 & 1 & \cdots & 1 \\ 0 & 1 & 1 & \cdots & 1 \\ 0 & 0 & 1 & \cdots & 1 \\ 0 & 0 & 0 & \cdots & 1 \\ & & \vdots & \ddots & \\ 0 & 0 & 0 & \cdots & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \\ (-1) \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 1 \\ \vdots \\ 1 \\ 1 \end{pmatrix}$$

Connection with the binomial coefficients $C\,(N,k)$

The same nonstrict upper unitriangular matrix becomes lower unitriangular when it is transposed and it has the special property that if we define its integer powers as the products by itself from the left side, those powers for the transpose viewed as an operator transform a column vector built with the unit everywhere into a sequence of nonzero binomial coefficients. The following equation illustrates this description.

$$\tilde{\mathbf{C}}\,(N,k) = \left\{ \begin{pmatrix} 1 & 1 & 1 & \cdots & 1 \\ 0 & 1 & 1 & \cdots & 1 \\ 0 & 0 & 1 & \cdots & 1 \\ 0 & 0 & 0 & \cdots & 1 \\ & & \vdots & \ddots & \\ 0 & 0 & 0 & \cdots & 1 \end{pmatrix}^{\mathbf{T}} \right\}^{\mathbf{k}} \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix} = \begin{pmatrix} C\,(k{+}0,k) \\ C\,(k{+}1,k) \\ C\,(k{+}2,k) \\ C\,(k{+}3,k) \\ \vdots \\ C\,(k{+}N{-}1,k) \end{pmatrix}$$

[4]All these results presented here were achieved after intensive reflections and experimental research done by the author since Nov 2012. Transcripted and reported to OEIS on Jan 31 2013. Acknowledgements to Maximilian Hasler, Antti Karttunen, and Alexander R. Povolotsky for their valuable help and moral support.